

# A Remark on CFT Realization of Quantum Doubles of Subfactors. Case Index $< 4$

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**ABSTRACT.** It is well-known that the quantum double  $D(N \subset M)$  of a finite depth subfactor  $N \subset M$ , or equivalently the Drinfeld center of the even part fusion category, is a unitary modular tensor category. Thus, it should arise in conformal field theory. We show that for every subfactor  $N \subset M$  with index  $[M : N] < 4$  the quantum double  $D(N \subset M)$  is realized as the representation category of a completely rational conformal net. In particular, the quantum double of  $E_6$  can be realized as a  $\mathbb{Z}_2$ -simple current extension of  $SU(2)_{10} \times \text{Spin}(11)_1$  and thus is not exotic in any sense. As a byproduct, we obtain a vertex operator algebra for every such subfactor.

We obtain the result by showing that if a subfactor  $N \subset M$  arises from  $\alpha$ -induction of completely rational nets  $\mathcal{A} \subset \mathcal{B}$  and there is a net  $\tilde{\mathcal{A}}$  with the opposite braiding, then the quantum  $D(N \subset M)$  is realized by completely rational net. We construct completely rational nets with the opposite braiding of  $SU(2)_k$  and use the well-known fact that all subfactors with index  $[M : N] < 4$  arise by  $\alpha$ -induction from  $SU(2)_k$ .

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## 1. INTRODUCTION

A unitary fusion category can be seen as the generalization of a finite group  $G$ , which is neither assumed to be commutative nor co-commutative. In particular, the easiest examples are the category  $\text{Rep}(G)$  of unitary representation of a finite group  $G$  and the category  $\text{Hilb}_G$  of  $G$ -graded finite dimensional Hilbert spaces. Note that  $G$  is co-commutative in the sense that  $\text{Rep}(G)$  is commutative, while  $G$  is in general non-commutative.

A factor is a von Neumann algebra with trivial center and a rather boring object. On the other hand a subfactor, an inclusion  $N \subset M$  of a factor  $N$  into another, turns often out to be a really interesting object. For example subfactor obtained by taking a fixed point with respect to a free action of a finite group  $N = M^G \subset M$  gives  ${}_M\mathcal{F}_M \cong \text{Hilb}_G$  and  ${}_N\mathcal{F}_N \cong \text{Rep}(G)$ . In general, a finite depth subfactor  $N \subset M$  gives two unitary fusion categories  ${}_N\mathcal{F}_N$  and  ${}_M\mathcal{F}_M$  which are (higher) Morita equivalent.

Conversely, having a unitary fusion category  $\mathcal{F}$ , there is a subfactor  $N \subset M$ , such that  ${}_N\mathcal{F}_N \cong {}_M\mathcal{F}_M \cong \mathcal{F}$ . An important invariant [Jon83] is the index  $[M : N]$  of a subfactor, which by Jones'

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*Date:* June 9, 2015.

Supported by NSF Grant DMS-1362138.

index theorem takes values in:

$$[M : N] \in \left\{ 4 \cos^2 \left( \frac{\pi}{m} \right) : m = 3, 4, \dots \right\} \cup [4 : \infty].$$

Another invariant is a pair of graphs, the principle and dual principal graphs, which are bipartite graphs. For index  $[M : N] < 4$  they are given by  $A$ - $D_{2n}$ - $E_{6,8}$  Dynkin diagrams, where the index is related to the Coxeter number  $m$  of the graph by  $[M : N] = 4 \cos^2(\pi/m)$ .

A unitary braided fusion category is a unitary fusion category with a braiding  $A$  braiding is a natural family of unitaries  $\varepsilon(\rho, \sigma) \in \text{Hom}(\rho \otimes \sigma, \sigma \otimes \rho)$ . Braided categories give a representation of the  $n$ -strand braid groups  $B_n = \langle e_1, \dots, e_{n-1} : e_{i+1}e_i e_{i+1} = e_i e_{i+1} e_i, e_i e_j = e_j e_i \text{ if } |i - j| \geq 2 \rangle$  on  $\text{Hom}(\rho^{\otimes n}, \rho^{\otimes n})$ . If  $\varepsilon(\rho, \sigma)\varepsilon(\sigma, \rho) = 1_{\sigma \otimes \rho}$  for all objects  $\sigma, \rho$ , it is called a symmetric fusion category. In this case the representations of the braid group are actually representations of the symmetric group. On the other hand, in a unitary modular tensor category (UMTC) the braiding is non-degenerated, in the sense that if  $\varepsilon(\rho, \sigma)\varepsilon(\sigma, \rho) = 1_{\sigma \otimes \rho}$  for all  $\rho$ , then  $\sigma$  is a direct sum of the trivial object.

Simple examples of UMTCs  $\mathcal{C}$  are the one where every irreducible object is invertible (has dimension 1). Then the fusion rules form an abelian group  $A$  and  $\mathcal{C}$  is characterized by a non-degenerated quadratic form on  $A$ .

The Drinfeld center of a UFC  $\mathcal{F}$ , or the quantum double of a finite depth subfactor  $N \subset M$ , which equals the Drinfeld center  $Z(\mathcal{F})$  of either of its fusion categories  $\mathcal{F} \in \{ {}_N\mathcal{F}_N, {}_M\mathcal{F}_M \}$ , is a unitary modular tensor category [Müg03b].

A coordinate version of modular tensor categories were invented by Moore and Seiberg [MS90] to axiomatize (the topological behaviour of) conformal field theories. Braided tensor categories also appeared in algebraic quantum field theory [FRS89] and UMTCs and their structure were analyzed by Rehren in [Reh90]. There are two axiomatizations for chiral CFT: vertex operator algebras (VOAs) and conformal nets and in both approaches the representation theory gives under certain sufficient conditions a (unitary) modular tensor category.

The natural question arises, if all modular tensor categories arise as representation categories of chiral CFT. A subquestion is if the quantum double of subfactors or equivalently Drinfeld centers of unitary fusion categories arise in this way.

We want to discuss such a question in the framework of conformal nets, which is naturally related to the study of subfactors. More precisely, if  $\mathcal{A}$  is a completely rational conformal net, then the category of Doplicher–Haag–Roberts representations  $\text{Rep}(\mathcal{A})$  is a unitary modular tensor category (UMTC) by [KLM01].

We vaguely conjecture that the following is true.

**Conjecture 1.1.** Let  $\mathcal{C}$  be a unitary modular tensor category (UMTC), then there is a completely rational conformal net  $\mathcal{A}$  with  $\text{Rep}(\mathcal{A}) \cong \mathcal{C}$ .

An analogous statement in higher dimensional algebraic quantum field theory (see [Haa96]) is known to be true. Namely, it is shown that under natural assumption for every net  $\mathcal{A}$  there is a compact (metrizable) group  $G$  with a central involutive element  $k \in G$ , such that the category of Doplicher–Haag–Roberts representations  $\text{DHR}(\mathcal{A})$  is the category of unitary representations of  $G$ , which is  $\mathbb{Z}_2$ -graded by  $k$ . Every such pair  $\{G, k\}$  can be realized using free field theory [DP02].

Conjecture 1.1 would imply the following weaker conjecture.

**Conjecture 1.2.** Let  $\mathcal{F}$  be a unitary fusion category (UFC), then there is a completely rational conformal net  $\mathcal{A}$  with  $\text{Rep}(\mathcal{A}) \cong Z(\mathcal{F})$ , where  $Z(\mathcal{F})$  is the Drinfeld center.

Equivalently, let  $N \subset M$  be a finite depth subfactor, then there is a completely rational conformal net  $\mathcal{A}$  with  $\text{Rep}(\mathcal{A}) \cong D(N \subset M)$ , where  $D(N \subset M)$  denotes the quantum double of  $N \subset M$ .

**Remark 1.3.** The net  $\mathcal{A}$  in Conjecture 1.1 or 1.2 would be far from unique. Namely, let  $\mathcal{B}$  be a holomorphic net, i.e. the representation category is trivial  $\text{Rep}(\mathcal{B}) \cong \text{Hilb}$ , then  $\text{Rep}(\mathcal{A} \otimes \mathcal{B}) \cong \text{Rep}(\mathcal{A})$ .

So far a technique which produces from a subfactor or a fusion category a conformal field theory is not established, though see [Jon14] for some recent approach. But subfactors up to index 5 are classified and we can try to exhaust (part of) the classification list, by constructing a CFT model for every subfactor in the list.

If we have a UMTC  $\mathcal{C}$  we can replace the braiding by its opposite braiding  $\varepsilon(\rho, \sigma) = \varepsilon(\sigma, \rho)^*$  which gives (in general) a new UMTC denoted  $\mathcal{C}^{\text{rev}}$ .

**Conjecture 1.4.** Let  $\mathcal{A}$  be a completely rational conformal net. Then there exist a completely rational conformal net  $\tilde{\mathcal{A}}$ , such that  $\text{Rep}(\tilde{\mathcal{A}}) \cong \text{Rep}(\mathcal{A})^{\text{rev}}$ .

Here the positivity of energy is crucial. One can easily construct  $\tilde{\mathcal{A}}$  with “negative energy” having this property. Note that Conjecture 1.4 would imply that Conjecture 1.2 would hold for all  $\mathcal{F} = \text{Rep}(\mathcal{A})$  which are representation category of a conformal net  $\mathcal{A}$ . Indeed,  $\mathcal{C} = \text{Rep}(\mathcal{A})$  is a UMTC and thus  $Z(\mathcal{C}) \cong \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} \cong \text{Rep}(\mathcal{A} \otimes \tilde{\mathcal{A}})$ .

There are more exotic subfactors for which the realization by conformal field theory in any sense is not known. The first is the Haagerup subfactor [Haa94]. Its quantum double is considered to be exotic in [HRW08]. In the same article also the quantum double of the  $E_6$  subfactor is considered exotic. The authors admit that they did not consider simple current extensions. We show that the double of  $E_6$  indeed just arises as  $\mathbb{Z}_2$ -simple current construction of  $\text{SU}(2)_{10} \times \text{SO}(11)_1$  and thus is far from exotic. We also note that the even part of the  $E_6$  subfactor is a pivotal fusion category of rank 3 and the lowest rank example of a pivotal fusion category which is not braided by the classification of rank 3 pivotal fusion categories [Ost13].

Conjecture 1.2 would give a positive answer to the question:

**Question 1.5.** Does every finite depth subfactor come from conformal field theory (cf. [Jon14]).

Namely, for every completely rational conformal net  $\mathcal{A}$ , Kawahigashi, Longo, Rehren and the author have recently shown that certain subfactors related to  $\text{Rep}(\mathcal{A})$  classify the phase boundaries of a full conformal field theory on Minkowski space based on the chiral theory  $\mathcal{A}$ .

**Proposition 1.6** (see Proposition 3.5). *Let  $N \subset M$  be a subfactor and a completely rational conformal net  $\mathcal{A}$  with  $\text{Rep}(\mathcal{A}) \cong D(N \subset M)$ . Then there is a phase boundary related to the subfactor  $N \subset M$ .*

So, in this sense Conjecture 1.2 would really give a positive answer to Question 1.5.

The main goal of this article is to confirm Conjecture 1.2 for the simple case  $[M : N] < 4$ .

**Proposition 1.7** (see Corollary 4.7). *Every quantum double  $D(N \subset M)$  of a subfactor  $N \subset M$  with  $[M : N] < 4$  is realized by a completely rational conformal net  $\mathcal{A}_{N \subset M}$ , i.e.  $\text{Rep}(\mathcal{A}_{N \subset M}) \cong D(N \subset M)$ .*

We note that the next possible index is realized by the Haagerup subfactor mentioned above with index

$$[M : N] = \frac{5 + \sqrt{13}}{2} \approx 4.303$$

and there is strong indication in [EG11], that there is a conformal net realizing its double. We hope that our techniques here give new ideas to construct this examples.

This article is organized as follows. In Section 2 we give some preliminaries about braided subfactors and quantum doubles and in Section 3 we give some preliminaries about conformal nets on the circle and introduce some examples which we later need. We give some characterization and structural results of conformal nets whose representation category is a quantum double. In Section 4 we give some results about conformal nets having the opposite braiding of a given net. We give

examples of nets having opposite braiding of  $SU(2)_k$ . We give a general criterion how the quantum double of a subfactor arising by  $\alpha$ -induction of an inclusion of conformal nets yields a conformal net realizing the quantum double of it. We use these techniques for the realization of quantum doubles for index less than 4 and some sporadic examples between 4 and 5. In Section 5, by using the categorical nature of our result, we show how to relate it to vertex operator algebras. In particular, there is also a realization of quantum doubles of subfactors with index less than 4 by vertex operator algebras.

**Acknowledgements.** I would like to thank Zhengwei Liu for some useful comment and Luca Giorgetti and Yasuyuki Kawahigashi for remarks on early version of this manuscript.

Ideas for this work were obtained at the Workshop ID: 1513 “Subfactors and Conformal Field Theory” at the Mathematische Forschungsinstitut Oberwolfach and the author would like to thank the organizers and the MFO.

## 2. QUANTUM DOUBLES

We are using here the language of endomorphisms of type III factors (see [BKLR15]), but the same can be understood in terms of bimodules of type II or type III factors or in terms of unitary fusion categories.

We note that it follows from [HY00], (more indirect also from [Pop95, Pop94a] and in certain cases [Ocn88]) that any abstract unitary fusion category  $\mathcal{F}$  can be realized as  $\mathcal{F} \subset \text{End}(M)$  with  $M$  the hyperfinite type III<sub>1</sub> factor. By Popa’s theorem [Pop95] such a realization is unique, namely if  $\tilde{\mathcal{F}} \subset \text{End}(N)$  another realization then there is an isomorphism  $N \rightarrow M$  implementing the equivalence between the two fusion categories (cf. [KLM01, Proof of Corollary 35]).

Given an inclusion  $N \subset M$  of hyperfinite type III<sub>1</sub> factors  $M, N$  with finite minimal index  $[M : N] < \infty$  [Jon83, Kos86] we denote by  $\iota: N \rightarrow M$  the inclusion map. We often write  $\iota(N) \subset M$  to have a uniform notation if we consider endomorphisms  $\rho$  of  $M$  and inclusions  $\rho(M) \subset M$ . By the finite index assumption, there is a conjugate morphism  $\bar{\iota}: M \rightarrow N$ , such that  $\text{id}_M \prec \iota \circ \bar{\iota}$  and  $\text{id}_N \prec \iota \circ \bar{\iota}$ . Then  $(\bar{\iota} \circ \iota)^{\circ n}$  and  $(\iota \circ \bar{\iota})^n$ ,  $n \in \mathbb{N}$  generate full  $C^*$ -tensor categories ([LR97])  ${}_N\mathcal{F}_N^\iota \subset \text{End}_0(N)$  and  ${}_M\mathcal{F}_M^{N \subset M} \subset \text{End}_0(N)$ , respectively, and we say that  $\iota(N) \subset M$  has finite depth if and only if  $|\text{Irr}({}_N\mathcal{F}_N^{N \subset M})| < \infty$ , or equivalently  $|\text{Irr}({}_M\mathcal{F}_M^{N \subset M})| < \infty$ . Similarly, one defines full replete subcategories  ${}_N\mathcal{F}_M^{N \subset M} = \langle (\bar{\iota} \circ \iota)^n \circ \bar{\iota} \rangle \subset \text{Mor}_0(M, N)$  and  ${}_M\mathcal{F}_N^{N \subset M} = \langle \iota \circ (\bar{\iota} \circ \iota)^n \rangle \subset \text{Mor}_0(N, M)$ .

The (strict) 2-category  $\mathcal{F}^{N \subset M}$  with two 0-objects  $\{N, M\}$  and the hom-categories given by  ${}_N\mathcal{F}_N^{N \subset M}$ ,  ${}_N\mathcal{F}_M^{N \subset M}$ ,  ${}_M\mathcal{F}_N^{N \subset M}$  and  ${}_M\mathcal{F}_M^{N \subset M}$  is called the **standard invariant** of  $N \subset M$ . The finite depth condition corresponds to rationality in conformal field theory.

Given a fusion category  ${}_N\mathcal{F}_N \subset \text{End}(N)$  and a subfactor  $N \subset M$  **related** to  ${}_N\mathcal{F}_N$ , i.e.  $\bar{\iota} \circ \iota \in {}_N\mathcal{F}_N$  (then  ${}_N\mathcal{F}_N^{N \subset M} \subset {}_N\mathcal{F}_N$ ) the **dual category**  ${}_M\mathcal{F}_M \subset \text{End}_0(M)$  is the fusion category generated by  $\beta \prec \iota \circ \rho \circ \bar{\iota}$  with  $\rho \in {}_N\mathcal{F}_N$ . The categories  ${}_N\mathcal{F}_N$  and  ${}_M\mathcal{F}_M$  are Morita equivalent in the sense of [Müg03a] the Morita equivalence is given by tensoring with  $\iota$  and  $\bar{\iota}$ .

We start with a unitary **modular tensor category** (UMTC)  ${}_N\mathcal{C}_N \subset \text{End}_0(N)$ , where the unitary braiding in  $\text{Hom}(\rho \circ \sigma, \sigma \circ \rho)$  is denoted by  $\varepsilon^+(\rho, \sigma)$  or simply  $\varepsilon(\rho, \sigma)$  and the reversed braiding by  $\varepsilon^-(\rho, \sigma) = \varepsilon(\sigma, \rho)^*$ .

Let us fix  $\iota(N) \subset M$  related to  ${}_N\mathcal{C}_N$ . This gives  $\theta = \bar{\iota} \circ \iota$  the structure of an algebra object in  ${}_N\mathcal{C}_N$ , more precisely a Q-system  $\Theta = (\theta, x, w)$ . There is a notion of commutativity, namely let  $x \in \text{Hom}(\theta, \theta \circ \theta)$  be the co-multiplication, then the Q-system is called **commutative** if and only if  $\varepsilon(\theta, \theta)x = x$ .

Let us fix a subfactor  $\iota(N) \subset M$  related to  ${}_N\mathcal{C}_N$ . Then  $\alpha$ -induction maps from  $\mathcal{C} = {}_N\mathcal{C}_N$  to the dual category  $\mathcal{D} = {}_M\mathcal{C}_M$  and is given by:

$$\begin{aligned} {}_N\mathcal{C}_N &\longrightarrow {}_M\mathcal{C}_M \subset {}_M\mathcal{C}_M \\ \lambda &\longmapsto \alpha_\lambda^\pm := \bar{\iota}^{-1} \circ \text{Ad}(\varepsilon^\pm(\lambda, \theta)) \circ \lambda \circ \bar{\iota} \in \text{End}(M). \end{aligned}$$

We denote by  $\mathcal{D}_\pm \equiv {}_M\mathcal{C}_M^\pm = \langle \alpha_\rho^\pm : \rho \in {}_N\mathcal{C}_N \rangle$  the UFC generated by  $\alpha^\pm$ -induction, respectively, and by  $\mathcal{D}_0 \equiv {}_M\mathcal{C}_M^0 = \mathcal{D}_+ \cap \mathcal{D}_-$  the **ambichiral** category.

Let  $\mathcal{F}$  be a unitary fusion category. We can assume that it is (essentially uniquely) realized as  $\mathcal{F} \cong {}_N\mathcal{F}_N \subset \text{End}_0(N)$  with  $N$  a hyperfinite type III<sub>1</sub> factor. Let  $A := N \otimes N^{\text{op}} \subset \iota_{\text{LR}}(B)$  be the Longo–Rehren inclusion with  ${}_A\mathcal{C}_A \cong {}_N\mathcal{F}_N \boxtimes {}_N\mathcal{F}_N^{\text{op}}$  and  ${}_B\mathcal{C}_B$  the category generated by  $\langle (\bar{\iota}_{\text{LR}} \circ \beta \circ \iota_{\text{LR}})^n : \beta \in {}_A\mathcal{C}_A \rangle \subset \text{End}(B)$ . Then Izumi showed that  ${}_B\mathcal{C}_B \cong Z(\mathcal{F})$ , where  $Z(\mathcal{F})$  denotes the unitary Drinfeld center [Müg03b, Section 6] of  $\mathcal{F}$ , which is a UMTC by [Müg03b]. The Q-system  $\Theta_{\text{LR}} = (\theta_{\text{LR}}, w_{\text{LR}}, x_{\text{LR}})$  with  $\theta = \bar{\iota}_{\text{LR}} \circ \iota_{\text{LR}}$  is commutative and  $d\theta_{\text{LR}} = \text{Dim}(\mathcal{F})$ , where  $\text{Dim}(\mathcal{F}) = \sum_{\rho \in \text{Irr}(\mathcal{F})} d\rho^2$  is the **global dimension**.

If we start with a finite depth subfactor  $\iota(N) \subset M$ , then  $Z({}_N\mathcal{F}_N^{N \subset M}) \cong Z({}_M\mathcal{F}_M^{N \subset M})$  (see proof of Proposition 2.2 below) and we can talk about the **quantum double** of  $\iota(N) \subset M$ , denoted by  $D(N \subset M)$ .

**Example 2.1.** The quantum double  $D(N \subset M)$  has been calculated in [EK98, Section 4] for  $A_n$  subfactors and [BEK01, Examples 5.1, 5.2] for  $E_6$  and  $E_8$  subfactors. The quantum double of  $E_6$  has also been computed using the tube algebra and half-braidings in [Izu01].

The quantum double is related to the Ocneanu's asymptotic inclusion [Ocn88], Popa's symmetric enveloping algebra [Pop94b] and the Longo–Rehren subfactor [LR95], see also [Mas00, Izu00].

Izumi showed [Izu00] that there is a Galois correspondence, namely there is a one-to-one correspondence between intermediate subfactors  $B \subset Q \subset A$  and subcategories  $\mathcal{G} \subset \mathcal{F}$ .

The following (3) was observed [Ocn01, Theorem 12] for  $\mathcal{C}$  being a  $\text{SU}(2)_k$  category and is partially contained in [BEK01, Corollary 3.10, 4.8].

**Proposition 2.2.** *Let  $\mathcal{C} \subset \text{End}(N)$  be a UMTC, and  $\iota(N) \subset M$  subfactor with commutative Q-system  $\Theta \in \mathcal{C}$ . Denote by  $\mathcal{D} = \langle \beta \prec \iota\rho\bar{\iota} : \rho \in \mathcal{C} \rangle \subset \text{End}(M)$  the dual category. Then*

- (1)  $Z(\mathcal{C}) \cong Z(\mathcal{D}) \cong \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ ,
- (2)  $Z(\mathcal{D}_0) \cong \mathcal{D}_0 \boxtimes \mathcal{D}_0^{\text{rev}}$ ,
- (3)  $Z(\mathcal{D}_+) \cong \mathcal{C} \boxtimes \mathcal{D}_0^{\text{rev}}$ ,
- (4)  $Z(\mathcal{D}_-) \cong \mathcal{C}^{\text{rev}} \boxtimes \mathcal{D}_0$ .

*Proof.* For (1) it follows by [Sch01] together with [Müg03b] that  $Z(\mathcal{C}) \cong Z(\mathcal{D})$ , because  $\mathcal{C}$  and  $\mathcal{D}$  are Morita equivalent, and again by [Müg03b]  $Z(\mathcal{C}) \cong \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ . It was shown e.g. in [BEK00, Theorem 4.2], that  $\mathcal{D}_0$  is modular, thus the statement (2) follows from (1).  $\mathcal{D}_+$  is equivalent with  $\mathcal{C}_\Theta$  (cf. [BKL14, Remark 5.6]) and by [DMNO13, Corollary 3.30], see also [DGNO10, Remark 4.3] we have  $Z(\mathcal{C}_\Theta) \cong \mathcal{C} \boxtimes \mathcal{C}_\Theta^{\text{rev}}$ , which is braided equivalent with  $\mathcal{C} \boxtimes \mathcal{D}_0$ , thus (3). Finally, (4) follows by applying (3) to  $\mathcal{C}^{\text{rev}}$ .  $\square$

### 3. CONFORMAL NETS

By a conformal net  $\mathcal{A}$ , we mean a local Möbius covariant net on the circle. It associates with every proper interval  $I \subset S^1 \subset \mathbb{C}$  on the circle a von Neumann algebra  $\mathcal{A}(I) \subset \text{B}(\mathcal{H}_\mathcal{A})$  on a fixed Hilbert space  $\mathcal{H}$ , such that the following properties hold:

- A. Isotony.**  $I_1 \subset I_2$  implies  $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$ .
- B. Locality.**  $I_1 \cap I_2 = \emptyset$  implies  $[\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$ .

- C. **Möbius covariance.** There is a unitary representation  $U$  of Möb on  $\mathcal{H}$  such that  $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)$ .
- D. **Positivity of energy.**  $U$  is a positive energy representation, i.e. the generator  $L_0$  (conformal Hamiltonian) of the rotation subgroup  $U(z \mapsto e^{i\theta}z) = e^{i\theta L_0}$  has positive spectrum.
- E. **Vacuum.** There is a (up to phase) unique rotation invariant unit vector  $\Omega \in \mathcal{H}$  which is cyclic for the von Neumann algebra  $\mathcal{A} := \bigvee_{I \in \mathcal{I}} \mathcal{A}(I)$ .

A local Möbius covariant net on  $\mathcal{A}$  on  $\mathbb{S}^1$  is called **completely rational** if it

- F. fulfills the **split property**, i.e. for  $I_0, I \in \mathcal{I}$  with  $\overline{I_0} \subset I$  the inclusion  $\mathcal{A}(I_0) \subset \mathcal{A}(I)$  is a split inclusion, namely there exists an intermediate type I factor  $M$ , such that  $\mathcal{A}(I_0) \subset M \subset \mathcal{A}(I)$ .
- G. is **strongly additive**, i.e. for  $I_1, I_2 \in \mathcal{I}$  two adjacent intervals obtained by removing a single point from an interval  $I \in \mathcal{I}$  the equality  $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)$  holds.
- H. for  $I_1, I_3 \in \mathcal{I}$  two intervals with disjoint closure and  $I_2, I_4 \in \mathcal{I}$  the two components of  $(I_1 \cup I_3)'$ , the  **$\mu$ -index** of  $\mathcal{A}$

$$\mu(\mathcal{A}) := [(\mathcal{A}(I_2) \vee \mathcal{A}(I_4))' : \mathcal{A}(I_1) \vee \mathcal{A}(I_3)]$$

(which does not depend on the intervals  $I_i$ ) is finite.

A **representation**  $\pi$  of  $\mathcal{A}$  is a family of representations  $\pi = \{\pi_I : \mathcal{A}(I) \rightarrow \mathcal{B}(\mathcal{H}_\pi)\}_{I \in \mathcal{I}}$  on a common Hilbert space  $\mathcal{H}_\pi$  which are compatible, i.e.  $\pi_J \upharpoonright \mathcal{A}(I) = \pi_I$  for  $I \subset J$ . Every non-degenerate representation  $\pi$  with  $\mathcal{H}_\pi$  separable turns—for every choice of an interval  $I_0 \in \mathcal{I}$ —out to be equivalent to a representation  $\rho$  on  $\mathcal{H}$ , such that  $\rho_J = \text{id}_{\mathcal{A}(J)}$  for  $J \cap I_0 = \emptyset$ . Then Haag duality implies that  $\rho_I$  is an endomorphism of  $\mathcal{A}(I)$  for every  $I \in \mathcal{I}$  with  $I \supset I_0$ . Thus we can realize the representation category of  $\mathcal{A}$  inside the  $C^*$  tensor category of endomorphisms  $\text{End}_0(N)$  of a type III factor  $N = \mathcal{A}(I)$  and the embedding turns out to be full and replete. We denote this category by  $\text{Rep}^I(\mathcal{A})$ . In particular, this gives the representations of  $\mathcal{A}$  the structure of a tensor category [DHR71]. It has a natural **braiding**, which is completely fixed by asking that if  $\rho$  is localized in  $I_1$  and  $\sigma$  in  $I_2$  where  $I_1$  is left of  $I_2$  inside  $I$  then  $\varepsilon(\rho, \sigma) = 1$  [FRS89]. The **statistical dimension** of  $\rho \in \text{Rep}^I(\mathcal{A})$  is given by  $d\rho = [N : \rho(N)]^{\frac{1}{2}}$ . Let  $\mathcal{A}$  be completely rational conformal net, then by [KLM01]  $\text{Rep}^I(\mathcal{A})$  is a UMTC and  $\mu_{\mathcal{A}} = \dim(\text{Rep}^I(\mathcal{A}))$ .

We write  $\mathcal{A} \subset \mathcal{B}$  or  $\mathcal{B} \supset \mathcal{A}$  if there is a representation  $\pi = \{\pi_I : \mathcal{A}(I) \rightarrow \mathcal{B}(I) \subset \mathcal{B}(\mathcal{H}_\mathcal{B})\}$  of  $\mathcal{A}$  on  $\mathcal{H}_\mathcal{B}$  and an isometry  $V : \mathcal{H}_\mathcal{A} \rightarrow \mathcal{H}_\mathcal{B}$  with  $V\Omega_\mathcal{A} = \Omega_\mathcal{B}$  and  $VU_\mathcal{A}(g) = U_\mathcal{B}(g)V$ . We ask that further that  $Va = \pi_I(a)V$  for  $I \in \mathcal{I}$ ,  $a \in \mathcal{A}(I)$ . Define  $p$  the projection on  $\mathcal{H}_{\mathcal{A}_0} = \overline{\pi_I(\mathcal{A}(I))\Omega}$ . Then  $pV$  is a unitary equivalence of the nets  $\mathcal{A}$  on  $\mathcal{H}_\mathcal{A}$  and  $\mathcal{A}_0$  defined by  $\mathcal{A}_0(I) = \pi_I(\mathcal{A}(I))p$  on  $\mathcal{H}_{\mathcal{A}_0}$ .

**Definition 3.1.** Let  $\mathcal{A} \subset \mathcal{B}$  an inclusion of conformal nets. Then we define the **coset net**  $\mathcal{A}^c(I) = \mathcal{B}(I) \cap \mathcal{A}'$ . Note that  $\mathcal{A}^c \subset \mathcal{B}$ . We call  $\mathcal{A} \subset \mathcal{B}$  **normal** if  $\mathcal{A}^{cc} = \mathcal{A}$ . We call  $\mathcal{A} \subset \mathcal{B}$  **co-finite** if  $[\mathcal{B}(I) : \mathcal{A}(I) \otimes \mathcal{A}^c(I)] < \infty$ .

For every co-finite extension  $\mathcal{A} \subset \mathcal{B}$  holds:  $\mathcal{B}$  is completely rational iff  $\mathcal{A}$  and  $\mathcal{A}^c$  are completely rational [Lon03].

**3.1. On conformal nets realizing quantum doubles/Drinfeld centers.** In this section we give some structural results about conformal nets whose representation category is a quantum double. If we talk about a subfactor  $N \subset M$ , we are just interested in finite depth subfactors which are hyperfinite of type  $\text{II}_1$  or  $\text{III}_1$ . In this case standard invariant is a complete invariant [Pop95]. We might also replace subfactor by subfactor standard invariant. We write  $N \subset M \approx N_1 \subset M_1$  if both have equivalent standard invariant.

**Definition 3.2.** A **holomorphic net**  $\mathcal{A}$  is a completely rational conformal net with trivial representation category  $\text{Rep}(\mathcal{A}) \cong \text{Hilb}$ , or equivalently [KLM01] with  $\mu_{\mathcal{A}} = 1$ .

**Proposition 3.3** (cf. [Müg10, Corollary 3.5], [Kaw15, Theorem 2.4]). *Let  $\mathcal{A}$  be a completely rational conformal net. The following are equivalent:*

- (1) *There is a holomorphic local irreducible extension  $\mathcal{B} \supset \mathcal{A}$ .*
- (2)  $\text{Rep}(\mathcal{A}) \cong Z(\mathcal{F})$  for some unitary fusion category  $\mathcal{F}$ .
- (3)  $\text{Rep}(\mathcal{A}) \cong D(N \subset M)$  for some finite depth subfactor  $N \subset M$ .

*Proof.* Given  $N \subset M$  take  $\mathcal{F} := {}_N\mathcal{F}_N$ . Conversely, we may assume that  $\mathcal{F}$  is a full subcategory of  $\text{End}(M)$  and we can take  $N = \rho(M) \subset M$ , where  $\rho = \bigoplus_{\rho_i \in \text{Irr}(\mathcal{F})} \rho_i$ . Thus (2) and (3) are equivalent.

If (2) is true the dual Q-system of the Longo–Rehren inclusion associated with  $\mathcal{F}$  gives a commutative Q-system  $\Theta = (\theta, w, x)$  in  $\text{Rep}^I(\mathcal{A})$  with  $d\theta = \sqrt{\mu_{\mathcal{A}}}$  the corresponding extension  $\mathcal{B} \supset \mathcal{A}$  has  $\mu_{\mathcal{B}} = 1$ .

Conversely, if (1) holds, let  $\Theta = (\theta, w, x)$  in  $\text{Rep}^I(\mathcal{A})$  be the Q-system characterizing  $\mathcal{B} \supset \mathcal{A}$ . The Q-system  $\Theta$  is commutative with  $d\theta = \sqrt{\text{Dim Rep } \mathcal{A}}$ , thus a Lagrangian Q-system which forces  $\text{Rep}(\mathcal{A}) \cong Z(\mathcal{F})$  for some fusion category  $\mathcal{F}$ .

Indeed, for  $N := \mathcal{A}(I) \subset \mathcal{B}(I) := M$  and  ${}_N\mathcal{C}_N = \text{Rep}^I(\mathcal{A})$  using Proposition 2.2 (3) we get

$$Z(\mathcal{F}) = Z({}_M\mathcal{C}_M^+) \cong {}_N\mathcal{C}_N \boxtimes {}_M\mathcal{C}_M^0 \cong \text{Rep}^I(\mathcal{A}) \boxtimes \text{Rep}^I(\mathcal{B}) \cong \text{Rep}^I(\mathcal{A})$$

using [BKL14, Proposition 6.4] in the second last step.  $\square$

**Remark 3.4.** One might see  $\mathcal{A} \subset \mathcal{B}$  as a generalization of an orbifold by a finite group. Namely, if  $\mathcal{F}$  is pointed and the fusion rules are given by the finite group  $G$ , then for the associated with  $\mathcal{F}$  associated extension  $\mathcal{B} \supset \mathcal{A}$  from Proposition 3.3 the net  $\mathcal{A} = \mathcal{B}^G$  is indeed the  $G$ -orbifold of  $\mathcal{B}$ , i.e.  $\mathcal{A} = \mathcal{B}^G$ , cf. [Müg05].

Let  $N \subset M$  be a finite index and finite depth subfactor. Conjecture 1.2 is equivalent with the existence of a conformal net  $\mathcal{A}$  with  $\text{Rep}(\mathcal{A}) \cong D(N \subset M)$  for every such  $N \subset M$ . Conversely, in the following Proposition we show that if such a net  $\mathcal{A}$  exists, there are two extensions  $\mathcal{B}_N$  and  $\mathcal{B}_M$ , such that  $\mathcal{B}_N(I) \subset \mathcal{B}_M(I) \approx N \subset M$ . But any morphism  $\beta: \mathcal{B}_N(I) \rightarrow \mathcal{B}_M(I)$  related to  $\text{Rep}^I(\mathcal{A})$ , i.e.  $\bar{\iota}_{\mathcal{B}_M(I)} \circ \beta \circ \iota_{\mathcal{B}_N(I)} \in \text{Rep}^I(\mathcal{A})$ , prescribes a defect line or phase boundary [BKLR14] between the full conformal field theories  $\mathcal{B}_L = \mathcal{B}_N \otimes \mathcal{B}_N \supset \mathcal{A} \otimes \mathcal{A}$  and  $\mathcal{B}_R \supset \mathcal{A} \otimes \mathcal{A}$  on 2D Minkowski space, which is invisible if restricted to  $\mathcal{A} \otimes \mathcal{A}$ , also called  $\mathcal{A}$ -topological. Here the net  $\mathcal{B}_R$  comes from the  $\alpha$ -induction construction [Reh00] of  $\mathcal{A} \subset \mathcal{B}_M$ , which coincides with the full center construction [BKL14]. Thus the subfactor  $\mathcal{B}_N(I) \subset \mathcal{B}_M(I) \approx N \subset M$  is related to a phase boundary in conformal field theory.

**Proposition 3.5.** *Let  $\mathcal{A}$  be a completely rational net with  $\text{Rep}(\mathcal{A}) \cong D(N \subset M)$ . Then there exist  $\mathcal{B}_N \supset \mathcal{A}$  local extension with  $\text{Rep}(\mathcal{B}_N) \cong \text{Hilb}$  and a (non-local) extension  $\mathcal{B}_M \supset \mathcal{B}_N \supset \mathcal{A}$  with  $\mathcal{B}_N(I) \subset \mathcal{B}_M(I) \approx N \subset M$ . Thus the inclusion  $\iota: \mathcal{B}_N(I) \rightarrow \mathcal{B}_M(I)$  is related to  $\text{Rep}^I(\mathcal{A})$  and prescribes a phase boundary in the sense of [BKLR14].*

*Proof.* The dual Q-systems of the Longo–Rehren inclusion associated with  ${}_N\mathcal{F}_N^{N \subset M}$  gives a commutative Q-systems in  $D(N \subset M) \cong Z({}_N\mathcal{F}_N^{N \subset M}) \cong Z({}_M\mathcal{F}_M^{N \subset M})$ , which we use to define the local extensions  $\mathcal{B}_N \supset \mathcal{A}$ . Let  $A = \mathcal{A}(I)$ ,  $B_N = \mathcal{B}_N(I)$ , then with  ${}_A\mathcal{C}_A \cong D(N \subset M)$  we have  ${}_B\mathcal{C}_{B_N} \cong {}_N\mathcal{F}_N^{N \subset M} \boxtimes ({}_N\mathcal{F}_N^{N \subset M})^{\text{op}}$ . Finally, the Q-system  $\Theta_{N \subset M} \boxtimes \text{id} = (\theta_{N \subset M} \boxtimes \text{id}, w_{N \subset M} \boxtimes 1_{\text{id}}, x_{N \subset M} \boxtimes 1_{\text{id}})$  gives an extension  $\mathcal{B}_M \supset \mathcal{B}_N$  which gives a non-local extension  $\mathcal{B}_M \supset \mathcal{A}$ , where  $\Theta_{N \subset M}$  is the Q-system in  ${}_N\mathcal{F}_N^{N \subset M}$  of the subfactor  $N \subset M$ . Because  $B_N \subset B_N$  and  $N \subset M$  have by construction equivalent Q-systems, they have the same standard invariant.  $\square$

### 3.2. Some conformal nets.

**Example 3.6.** We denote by  $\mathcal{A}_{\text{SU}(2),k}$  or simply by  $\mathcal{A}_k$  the  $\text{SU}(2)$  loop group net at level  $k$  [Was98], which is completely rational [Xu00] and thus gives a UTMC  $\text{Rep}(\mathcal{A}_k)$ . The simple objects are  $\{\rho_0, \dots, \rho_k\}$  with fusion rules

$$[\rho_i] \times [\rho_j] = \bigoplus_{\substack{\ell=|i-j| \\ i+\ell \text{ even} \\ i+j+\ell \leq 2k}}^{i+j} [\rho_\ell].$$

The dimensions  $d\rho_i$  and twists  $\omega_{\rho_i}$  are given by

$$d_i = d\rho_i = [i+1]_q := \frac{\sin \frac{(i+1)\pi}{k+2}}{\sin \frac{\pi}{k+2}}, \quad \omega_i = \omega_{\rho_i} = \exp\left(2\pi i \frac{i(i+2)}{4(k+2)}\right), \quad q = \exp\left(\frac{i\pi}{k+2}\right)$$

and the central charge  $c_k$  and global dimension  $D_k$  by

$$c_k = \frac{3k}{k+2}, \quad D_k = \sum_{i=0}^k d_i^2 = \frac{k+2}{2 \sin^2\left(\frac{\pi}{k+2}\right)}.$$

We remember the classification of  $\text{SU}(2)_k$  conformal nets [KL04], [BE98].

**Proposition 3.7.** *Local irreducible extensions  $\mathcal{B} \supset \mathcal{A}_k$ , i.e. a local net  $\mathcal{B}$  containing  $\mathcal{A}_k$  as a subnet, such that  $\mathcal{A}_k(I) \cap \mathcal{B}(I) = \mathbb{C}$  are in one-to-one correspondence with  $A$ - $D_{2n}$ - $E_{6,8}$  Dynkin diagrams of Coxeter number  $k+2$ . The  $E_{6,8}$  Dynkin diagram correspond to the conformal inclusions  $\mathcal{A}_{10} \subset \mathcal{A}_{\text{Spin}(5),1}$  and  $\mathcal{A}_{28} \subset \mathcal{A}_{G_2,1}$ , respectively.*

*The subfactor  $\alpha_{\rho_1}^\pm(\mathcal{B}(I)) \supset \mathcal{B}(I)$  has a principal graph the corresponding Dynkin diagram.*

**Example 3.8.** The loop group net of  $\text{Spin}(2n+1)$  at level 1  $\mathcal{A}_{\text{Spin}(2n+1),1}$  [Böc96, Theorem 3.1] and [Xu09, Lemma 3.1] has global dimension  $D = 4$  and has the Ising fusion rules, i.e. the same fusion rules as the net  $\mathcal{A}_{\text{SU}(2),2} = \mathcal{A}_{\text{Spin}(3),1}$ . We denote the (choice of simple objects) by  $\{\rho_0, \rho_1, \rho_2\}$ . The category is determined by the fusion rules and twists [FK93, Proposition 8.2.6], which are:

$$\omega_{\rho_1} = \exp\left(\frac{2\pi i(2n+1)}{16}\right), \quad \omega_{\rho_2} = -1.$$

**Example 3.9.** We get a net  $\mathcal{A}_{G_2,1}$  associated with  $(G_2)_1$  as an extension of  $\mathcal{A}_{28}$ . The category of representations is the Fibonacci or golden category with fusion rules  $[\tau] \times [\tau] = [\text{id}] + [\tau]$ .

There is a conformal inclusion of  $\mathcal{A}_{\text{SU}(3),2} \otimes \mathcal{A}_{\text{SU}(3),1} \subset \mathcal{A}_{F_4,1}$ , thus  $\mathcal{A}_{F_4,1}$  is completely rational. There is also  $\mathcal{A}_{F_4,1} \otimes \mathcal{A}_{G_2,1} \subset \mathcal{A}_{E_8,1}$ , in particular  $\text{Rep}(\mathcal{A}_{G_2,1}) \cong \text{Rep}(\mathcal{A}_{F_4,1})^{\text{rev}}$ , which is an application of Proposition 4.3.

**Example 3.10.** For central charge  $c$  with values

$$c_m = 1 - \frac{6}{(m+1)(m+2)}, \quad (m = 2, 3, \dots),$$

the **Virasoro net**  $\text{Vir}_{c_m}$  is given by the coset net of the inclusion  $\mathcal{A}_m \subset \mathcal{A}_{m-1} \otimes \mathcal{A}_1$  [KL04], in other words we have the conformal inclusion

$$\text{Vir}_{c_m} \otimes \mathcal{A}_m \subset \mathcal{A}_{m-1} \otimes \mathcal{A}_1.$$

and  $\text{Vir}_{c_m}$  is completely rational, see [KL04].

## 4. REALIZATION OF SOME QUANTUM DOUBLES BY CONFORMAL NETS

## 4.1. Realization of the opposite braiding.

**Proposition 4.1.** *Let  $\mathcal{A}, \tilde{\mathcal{A}}$  be completely rational conformal nets with  $\text{Rep}(\tilde{\mathcal{A}}) \cong \text{Rep}(\mathcal{A})^{\text{rev}}$  and  $\mathcal{B} \supset \mathcal{A}$  be an irreducible local extension (which is automatically completely rational). Then there is an irreducible local extension  $\tilde{\mathcal{B}} \supset \tilde{\mathcal{A}}$  with  $\text{Rep}(\tilde{\mathcal{B}}) \cong \text{Rep}(\mathcal{B})^{\text{rev}}$ .*

*Proof.* Using the equivalence  $\text{Rep}(\tilde{\mathcal{A}}) \cong \text{Rep}(\mathcal{A})^{\text{rev}}$ , the commutative Q-system  $\Theta \in \text{Rep}(\mathcal{A})$  gives a commutative Q-system  $\tilde{\Theta} \in \text{Rep}(\tilde{\mathcal{A}})$ , which defines an extension  $\tilde{\mathcal{B}} \supset \tilde{\mathcal{A}}$  with the asked properties.  $\square$

*Remark 4.2.* This is a trivial instance of mirror extensions [Xu07], namely take  $\mathcal{B}_{\text{LR}} \supset \mathcal{A} \otimes \tilde{\mathcal{A}}$  the Longo–Rehren extension [LR95], which gives  $\text{Rep}(\mathcal{B}_{\text{LR}}) \cong \text{Hilb}$ . Then  $\mathcal{A} \subset \mathcal{B}_{\text{LR}}$  is normal and co-finite and  $\tilde{\mathcal{A}}$  is its coset and  $\tilde{\mathcal{B}} \supset \tilde{\mathcal{A}}$  the mirror extension of  $\mathcal{A} \subset \mathcal{B}$ . Using [BKL14, Proposition 6.4]  $\text{Rep}(\tilde{\mathcal{B}})$  is equivalent as UFC with  $\text{Rep}(\mathcal{B})$  and has the opposite braiding.

**Proposition 4.3.** *Let  $\mathcal{B}$  be a holomorphic net and  $\mathcal{A} \subset \mathcal{B}$  be co-finite and normal. Let  $\tilde{\mathcal{A}}$  be the coset net of the inclusion  $\mathcal{A} \subset \mathcal{B}$ . Then the nets  $\mathcal{A}$  and  $\mathcal{A}^c$  are completely rational with  $\text{Rep}(\mathcal{A}^c) \cong \text{Rep}(\mathcal{A})^{\text{rev}}$ .*

*Proof.*  $\mathcal{A}$  and  $\mathcal{A}^c$  are completely rational by assumption (using [Lon03] see above).

The Q-system  $\Theta = (\theta, w, x)$  giving the extension  $\mathcal{A} \otimes \mathcal{A}^c \subset \mathcal{B}$  is of the form

$$[\theta] = \bigoplus_{\substack{\mu \in \text{Irr}(\text{Rep}(\mathcal{A})) \\ \nu \in \text{Irr}(\text{Rep}(\mathcal{A}^c))}} Z_{\mu, \nu} [\mu \otimes \nu].$$

By normality of  $\mathcal{A}, \mathcal{A}^c \subset \mathcal{B}$  we have  $Z_{\mu, \text{id}} = \delta_{\text{id}, \mu}$  and  $Z_{\text{id}, \nu} = \delta_{\text{id}, \nu}$ . Then it follows that there is a braided equivalence  $\phi: \mathcal{C} \rightarrow \mathcal{D}^{\text{rev}}$ , for some full and replete subcategories  $\mathcal{C} \subset \text{Rep}(\mathcal{A})$  and  $\mathcal{D} \subset \text{Rep}(\mathcal{B})$ , such that  $\Theta$  is the by  $\phi$  twisted Longo–Rehren extension, see [BKL14, Definition 4.1] for the definition. On the one hand  $(d\theta)^2 = \text{Dim } \text{Rep}(\mathcal{A}) \cdot \text{Dim } \text{Rep}(\mathcal{A}^c)$ , because  $\mathcal{B}$  is holomorphic. On the other hand  $d\theta = \text{Dim } \mathcal{C} = \text{Dim } \mathcal{D}$ . Together, because all dimensions are positive, this implies  $\mathcal{C} = \text{Rep}(\mathcal{A})$  and  $\mathcal{D} = \text{Rep}(\mathcal{A}^c)$ .  $\square$

Let  $\mathcal{A}_k = \mathcal{A}_{\text{SU}(2), k}$  and let  $\mathcal{B}_k$  be the coset net of

$$\mathcal{A}_k \subset \mathcal{A}_1^{\otimes k} = \mathcal{A}_1^{\otimes k}$$

which is normal by [Xu07, Lemma 4.2 (1)]. By induction, it follows that we have conformal inclusions:

$$\mathcal{A}_k \otimes \text{Vir}_{c_2} \otimes \cdots \otimes \text{Vir}_{c_k} \subset \mathcal{A}_k \otimes \mathcal{B}_k \subset \mathcal{A}_{\text{SU}(2), 1}^{\otimes k}$$

thus  $\mathcal{B}_k$  it is completely rational by [Lon03]. Using the conformal inclusion  $\mathcal{A}_{E_7, 1} \otimes \mathcal{A}_1 \subset \mathcal{A}_{E_8, 1}$ , which are all conformal nets associated with even lattices (cf. [Bis12]) and which is a Longo–Rehren extension and thus normal we get the conformal inclusions:

$$\mathcal{A}_k \otimes \mathcal{B}_k \otimes \mathcal{A}_{E_7, 1}^{\otimes k} \subset \mathcal{A}_{A_1}^{\otimes k} \otimes \mathcal{A}_{E_7, 1}^{\otimes k} \subset \mathcal{A}_{E_8, 1}^{\otimes k}.$$

Now we take  $\tilde{\mathcal{A}}_k$  to be the coset of the normal inclusion [Xu07, Lemma 4.2 (1)]  $\mathcal{A}_k \subset \mathcal{A}_{E_8, 1}^k$ . This is completely rational, because it is an intermediate net of completely rational nets:

$$\mathcal{A}_k \otimes \mathcal{B}_k \otimes \mathcal{A}_{E_7, 1}^{\otimes k} \subset \mathcal{A}_k \otimes \tilde{\mathcal{A}}_k \subset \mathcal{A}_{E_8, 1}^{\otimes k}.$$

Thus using Proposition 4.3 we have proven:

**Proposition 4.4.** *The coset net  $\tilde{\mathcal{A}}_k$  of the inclusion  $\mathcal{A}_k \subset \mathcal{A}_{E_8, 1}^k$  above is completely rational with  $\text{Rep}(\tilde{\mathcal{A}}_k) \cong \text{Rep}(\mathcal{A}_k)^{\text{rev}}$ .*

**Example 4.5.** We note that  $\tilde{\mathcal{A}}_1 = \mathcal{A}_{E_7,1}$  and that  $\text{Vir}_{c_k} \otimes \tilde{\mathcal{A}}_1 \otimes \tilde{\mathcal{A}}_{k-1} \otimes \mathcal{A}_k \subset \mathcal{A}_{E_8,1}^{\otimes k}$ . We get the intermediate inclusion:

$$\text{Vir}_{c_k} \otimes \tilde{\mathcal{A}}_1 \otimes \tilde{\mathcal{A}}_{k-1} \otimes \mathcal{A}_k \subset \tilde{\mathcal{A}}_k \otimes \mathcal{A}_k \subset \mathcal{A}_{E_8,1}^{\otimes k}$$

Thus also  $\text{Vir}_{c_k} \otimes \tilde{\mathcal{A}}_{k-1} \otimes \tilde{\mathcal{A}}_1 \subset \tilde{\mathcal{A}}_k$  and  $\text{Vir}_{c_k}$  can be obtained back from the coset of  $\tilde{\mathcal{A}}_{k-1} \subset \tilde{\mathcal{A}}_1 \otimes \tilde{\mathcal{A}}_k$ . We also get that  $\text{Vir}_{c_m} \subset \mathcal{A}_{E_8,1}^{\otimes m}$  is normal and co-finite, and thus its coset  $\tilde{\text{Vir}}_{c_m} = \text{Vir}_{c_m}^c$  realizes the opposite braiding of  $\text{Vir}_{c_m}$ . Further,  $\text{Vir}_{c_m} \otimes \tilde{\text{Vir}}_{c_m}$  realizes, using Proposition 2.2(1), the Drinfeld center  $Z(\text{Rep}^I(\text{Vir}_{c_m}))$ .

**4.2. Realization of quantum doubles.** The next proposition shows, that if a subfactor  $N \subset M$  arises from  $\alpha$ -induction of a local irreducible extension  $\mathcal{A} \subset \mathcal{B}$  and we have a net  $\tilde{\mathcal{A}}$  realizing the opposite braiding of  $\mathcal{A}$ , then there is a net  $\mathcal{B}_{N \subset M}$  with  $\text{Rep}(\mathcal{B}_{N \subset M}) = \mathcal{D}(N \subset M)$ .

**Proposition 4.6.** *Let  $N \subset M$  be an irreducible subfactor. Assume there exists a completely rational conformal net  $\mathcal{A}$  and an irreducible local extension  $\mathcal{B} \supset \mathcal{A}$ , such that  $N \subset M$  arises by  $\alpha^\pm$  induction, i.e. there is a  $\rho \in {}_{\mathcal{A}(I)}\mathcal{C}_{\mathcal{A}(I)} = \text{Rep}^I(\mathcal{A})$  and a  $[\beta] \prec [\alpha_\rho^\pm]$ , such that  $\beta(\mathcal{B}(I)) \subset \mathcal{B}(I) \sim N \subset M$ . Further, assume there exists  $\tilde{\mathcal{A}}$ , a completely rational conformal net with  $\text{Rep}(\tilde{\mathcal{A}}) \cong \text{Rep}(\mathcal{A})^{\text{rev}}$ . Then*

- (1) *There exists a completely rational conformal net  $\mathcal{B}_{N \subset M}$  realizing the quantum double  $\mathcal{D}(N \subset M)$ , i.e.  $\text{Rep}(\mathcal{B}_{N \subset M}) \cong \mathcal{D}(N \subset M)$ .*
- (2) *It can be given as a local irreducible extension:*
  - $\mathcal{B}_{N \subset M} \supset \mathcal{A} \otimes \tilde{\mathcal{B}}$ , in the  $\alpha^+$  case or
  - $\mathcal{B}_{N \subset M} \supset \tilde{\mathcal{A}} \otimes \mathcal{B}$  in the  $\alpha^-$  case.
- (3) *In the case that  $[\beta], [\bar{\beta}]$  (tensor) generate  ${}_{\mathcal{B}(I)}\mathcal{C}_{\mathcal{B}(I)}^\pm$ , but  $[\bar{\beta} \circ \beta]$  does not, the former extension is a  $\mathbb{Z}_2$ -simple current extension.*
- (4) *In the case that  $[\bar{\beta} \circ \beta]$  (tensor) generates  ${}_{\mathcal{B}(I)}\mathcal{C}_{\mathcal{B}(I)}^\pm$ , then  $\mathcal{B}_{N \subset M}$  equals  $\mathcal{A} \otimes \tilde{\mathcal{B}}$  or  $\tilde{\mathcal{A}} \otimes \mathcal{B}$ , respectively.*

*Proof.* By Proposition 2.2 we have  $\text{Rep}(\mathcal{A} \otimes \tilde{\mathcal{B}}) \cong Z({}_{\mathcal{B}(I)}\mathcal{C}_{\mathcal{B}(I)}^+)$  and  $\text{Rep}(\tilde{\mathcal{A}} \otimes \mathcal{B}) \cong Z({}_{\mathcal{B}(I)}\mathcal{C}_{\mathcal{B}(I)}^-)$ .

Let  ${}_{\mathcal{B}(I)}\mathcal{C}_{\mathcal{B}(I)}^\beta \subset {}_{\mathcal{B}(I)}\mathcal{C}_{\mathcal{B}(I)}^\pm$  be the subcategory (tensor) generated by  $\bar{\beta} \circ \beta$ , then by  $\mathcal{D}(N \subset M) \cong Z({}_{\mathcal{B}(I)}\mathcal{C}_{\mathcal{B}(I)}^\beta)$  by assumption.

Further, there is a holomorphic net  $\mathcal{B}_{\text{holo}} \supset \mathcal{A} \otimes \tilde{\mathcal{B}}$  or  $\mathcal{B}_{\text{holo}} \supset \tilde{\mathcal{A}} \otimes \mathcal{B}$ , respectively, which is the Longo–Rehren inclusion and by Galois correspondence there is an intermediate net  $\mathcal{B}_{N \subset M}$  with  $\text{Rep}(\mathcal{B}_{N \subset M}) \cong Z({}_{\mathcal{B}(I)}\mathcal{C}_{\mathcal{B}(I)}^\beta) \cong \mathcal{D}(N \subset M)$ .

In the case of (2) we have  $2 \text{Dim } {}_{\mathcal{B}(I)}\mathcal{C}_{\mathcal{B}(I)}^\beta = \text{Dim } {}_{\mathcal{B}(I)}\mathcal{C}_{\mathcal{B}(I)}^\pm$  and  $\mathcal{B}_{\text{holo}} \supset \mathcal{A} \otimes \tilde{\mathcal{B}}$  or  $\mathcal{B}_{\text{holo}} \supset \tilde{\mathcal{A}} \otimes \mathcal{B}$ , respectively, have index two, thus it is a  $\mathbb{Z}_2$ -simple current extensions.

In the case of (3) we have  ${}_{\mathcal{B}(I)}\mathcal{C}_{\mathcal{B}(I)}^\beta = {}_{\mathcal{B}(I)}\mathcal{C}_{\mathcal{B}(I)}^\pm$ , respectively and the extension is trivial.  $\square$

For subfactors with index  $< 4$  it is well-known that they arise via  $\alpha$ -induction from  $\text{SU}(2)_k$  loop group models  $\mathcal{A}_k$ , see Proposition 3.7. Together with  $\tilde{\mathcal{A}}_k$  from Proposition 4.4 we thus get:

**Corollary 4.7.** *For every subfactor  $N \subset M$  with  $[M : N] < 4$ , i.e. for every the standard invariant label by  $G \in \{A_n, D_{2n}, E_{6,8}, \bar{E}_{6,8}\}$ , there is a conformal net  $\mathcal{A}_{N \subset M}$  with  $\text{Rep}(\mathcal{A}_{N \subset M}) = \mathcal{D}(N \subset M)$ . The realizations can be given as follows:*

$A_{k+1}$ :  $\mathcal{A}_k \otimes \tilde{\mathcal{A}}_k \rtimes_{\rho_{k,k}} \mathbb{Z}_2$  the simple current extension with respect to the automorphism  $\rho_k \otimes \tilde{\rho}_k$ .

$D_{2n}$ :  $\mathcal{B}_{D_{2n}} \otimes \tilde{\mathcal{B}}_{D_{2n}}$ , where  $\mathcal{B}_{D_{2n}}$  and  $\tilde{\mathcal{B}}_{D_{2n}}$  are the  $\mathbb{Z}_2$ -simple current extensions of  $\mathcal{A}_{4n-4}$  and  $\tilde{\mathcal{A}}_{4n-4}$  by  $\rho_{4n-4}$  and  $\tilde{\rho}_{4n-4}$ , respectively.

$E_6$ :  $\mathcal{A}_{10} \otimes \mathcal{A}_{\text{Spin}(11),1} \rtimes_{[\rho_{10,2}]} \mathbb{Z}_2$ , where we can replace  $\mathcal{A}_{\text{Spin}(11),1}$  by  $\tilde{\mathcal{B}} \supset \tilde{\mathcal{A}}_{10}$ , the extension obtained from Proposition 4.1 applied to  $\mathcal{A}_{10} \subset \mathcal{A}_{\text{Spin}(5),1}$ .

$\bar{E}_6$ :  $\tilde{\mathcal{A}}_{10} \times \mathcal{A}_{\text{Spin}(5),1} \rtimes_{[\rho_{10,2}]} \mathbb{Z}_2$ .

$E_8$ :  $\mathcal{A}_{28} \otimes \mathcal{A}_{F_4,1} \rtimes_{[\rho_{28,0}]} \mathbb{Z}_2$ , i.e. it is given by  $\mathcal{B}_{D_{16}} \otimes \mathcal{A}_{F_4,1}$ . We can replace  $\mathcal{A}_{F_4,1}$  by the  $\tilde{\mathcal{B}} \supset \tilde{\mathcal{A}}_{28}$ , the extension obtained from Proposition 4.1 applied to  $\mathcal{A}_{28} \subset \mathcal{A}_{G_2,1}$ .

$\bar{E}_8$ :  $\tilde{\mathcal{A}}_{28} \otimes \mathcal{A}_{G_2,1} \rtimes_{[\rho_{28,0}]} \mathbb{Z}_2$  i.e. it is given by  $\tilde{\mathcal{B}}_{D_{16}} \otimes \mathcal{A}_{G_2,1}$ .

*Proof.* All subfactors arise as  $\alpha_{\rho_1}^\pm(\mathcal{B}_G(I)) \subset \mathcal{B}_G(I)$ , where  $\mathcal{B}_G \supset \mathcal{A}_k$  is the extension in Proposition 3.7. Further  $\tilde{\alpha}_{\rho_1}^\pm$  generates  ${}_{\mathcal{B}(I)}\mathcal{C}_{\mathcal{B}(I)}^\pm$ , while  $\tilde{\alpha}_{\rho_1}^\pm \circ \alpha_{\rho_1}^\pm$  does not. Thus in each case we are in the situation of case (2) of Proposition 4.6 and in each case there is just one possible  $\mathbb{Z}_2$ -simple current extension.  $\square$

*Remark 4.8.* Our method also applies to some subfactors with index between 4 and 5:

- The GHJ subfactor [GdlHJ89] with index  $3 + \sqrt{3}$  arises as the subfactor  $\mathcal{A}_{10}(I) \subset \mathcal{A}_{\text{Spin}(5),1}(I)$ , see [BEK99, Section 2.2]. Thus the even part of it coincides with the even part of  $\text{Rep}(\mathcal{A}_{10})$ , i.e. with the even part of the  $A_{11}$  subfactor. Thus its quantum double is the same as of the  $A_{11}$  subfactor and thus also realized by  $\mathcal{A}_{10} \otimes \tilde{\mathcal{A}}_{10} \rtimes_{[\rho_{10,10}]} \mathbb{Z}_2$ .
- The 2221 subfactor with index  $(5 + \sqrt{21})/2$  arises from the conformal inclusion  $\mathcal{A}_{G_2,3} \subset \mathcal{A}_{E_6,1}$  by  $\alpha$ -induction [Xu01], see also [CMS11, Appendix]. The subfactor was also constructed by Izumi in [Izu00]. Note that  $\text{Rep}(\mathcal{A}_{SU(3),1}) \cong \text{Rep}(\mathcal{A}_{E_6,1})^{\text{rev}}$ , thus by Proposition 4.6 (3) the net  $\mathcal{A}_{G_2,3} \otimes \mathcal{A}_{SU(3),1}$  realizes its quantum double. A similar observation was made by Ostrik [CMS11, Remark A.4.3]. The complex conjugate should be realized by  $\tilde{\mathcal{A}}_{G_2,3} \otimes \mathcal{A}_{E_6,1}$ , but we do not know how to realize the net  $\tilde{\mathcal{A}}_{G_2,3}$ .

**4.3. Modular invariants.** All our examples in Corollary 4.7 are  $\mathbb{Z}_2$ -simple current extension. We remember that for  $\mathcal{A} \subset \mathcal{B}$  an extension,  $N = \mathcal{A}(I) \subset \mathcal{B}(I) = M$  and  ${}_N\mathcal{C}_N = \text{Rep}^I(\mathcal{A})$ , the matrix  $Z = (Z_{\mu,\nu})_{\mu,\nu \in \text{Irr}({}_N\mathcal{C}_N)}$  with  $Z_{\mu,\nu} = \dim \text{Hom}(\alpha_\mu^+, \alpha_\nu^-)$  is a modular invariant [BEK99], i.e. commutes with the  $S$  and  $T$  associated with  ${}_N\mathcal{C}_N$ . The modular invariant of a commutative  $\mathbb{Z}_2$ -simple current extension  $\theta = [\rho_0] \oplus [\rho_g]$  is given by (cf. (3.59) in [FRS04] for the general formula)

$$Z_{i,j} = \frac{1}{2} \left( 1 + \frac{\omega_{gi}}{\omega_i} \right) (\delta_{i,j} + \delta_{gi,j}),$$

where  $gi$  is the action of  $g$  on  $i$ , i.e.  $[\rho_{gi}] = [\rho_g] \times [\rho_i]$ . We conveniently write the modular invariant in character form as:

$$Z = \sum_{\mu,\nu} Z_{\mu,\nu} \chi_\mu \bar{\chi}_\nu.$$

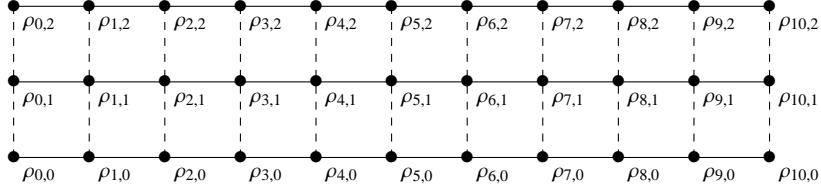
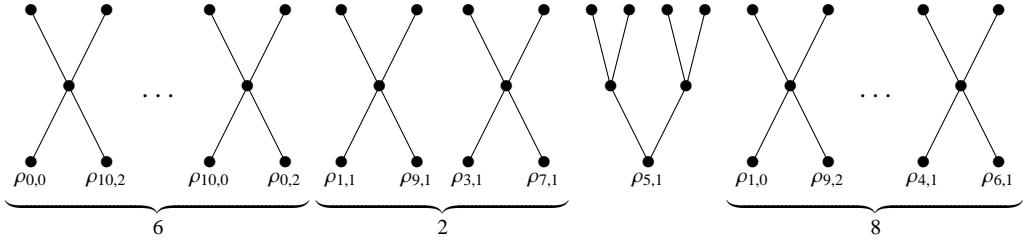
We include the modular invariants, from which one can derive the fusion rules of the representation category. We note, although it is not necessary and follows from the above “abstract non-sense”, one can directly check that the, for example the representation category of the net  $\mathcal{A}_{10} \otimes \mathcal{A}_{\text{Spin}(11),1} \rtimes_{[\rho_{10,2}]} \mathbb{Z}_2$  has the fusion rules of the  $E_6$  double as in [Izu01, HRW08]. Some of this calculation is contained in [BEK01].

**Example 4.9** ( $A_{k+1}$ -case). For the inclusion  $\mathcal{A}_k \otimes \tilde{\mathcal{A}}_k \subset \mathcal{A}_{N \subset N} = (\mathcal{A}_k \otimes \tilde{\mathcal{A}}_k) \rtimes \mathbb{Z}_2$  has the modular invariant is given by:

$$Z_{\rho_{i_1,j_1}, \rho_{i_2,j_2}} = \frac{1}{2} \left( 1 + (-1)^{i_1-j_1} \right) (\delta_{i_1,i_2} \delta_{j_1,j_2} + \delta_{i_1,k-i_2} \delta_{j_1,k-j_2})$$

and thus

$$Z = \frac{1}{2} \sum_{\substack{i,j=0 \\ i+j=\text{even}}}^k |\chi_{\rho_{i,j}} + \chi_{\rho_{k-i,k-j}}|^2.$$

FIGURE 1. Fusion rules of  $SU(2)_{10} \times \text{Spin}(11)_1$ FIGURE 2. (Dual) principal graph for  $A_{SU(2),10} \otimes A_{\text{Spin}(11),1} \subset A_{N \subset M}$ 

**Example 4.10** ( $D_{2n}$ -case). Let  $k = 4n - 4$ . Let  $\mathcal{A}_k$ , then there is a simple current extension  $\mathcal{B}_k = \mathcal{A}_k \rtimes_{\rho_k} \mathbb{Z}_2$  of  $\mathcal{A}_k$  corresponding to the Dynkin diagram  $D_{2n}$  in Proposition 3.7 with modular invariant:

$$Z_{D_{2n}} = \frac{1}{2} \sum_{\ell=0}^{\frac{k}{2}} |\chi_{2\ell} + \chi_{k-2\ell}|^2.$$

The same is true for  $\tilde{\mathcal{B}}_k = \tilde{\mathcal{A}}_k \rtimes_{\rho_k} \mathbb{Z}_2$ . The net  $\mathcal{A}_{N \subset M}$  for  $D_{2n}$  is just  $\mathcal{B}_k \otimes \tilde{\mathcal{B}}_k$ , which is an  $\mathbb{Z}_2$  extension of

$$\mathcal{A}_k \otimes \tilde{\mathcal{B}}_k \subset \mathcal{B}_k \otimes \tilde{\mathcal{B}}_k \supset \mathcal{B}_k \otimes \tilde{\mathcal{A}}_k.$$

So the modular invariant for the  $\mathbb{Z}_2$ -simple current extension is  $Z_{D_{2n}} \otimes I_{n+1}$ , where  $I_m$  is the  $m \times m$  identity matrix.

**Example 4.11** ( $E_6$ -cases). Then modular invariant for  $A_{SU(2),10} \otimes A_{\text{Spin}(11),1} \subset A_{N \subset M}$  for  $E_6$  is given by:

$$Z = X + Y + 2|\chi_{5,1}|^2,$$

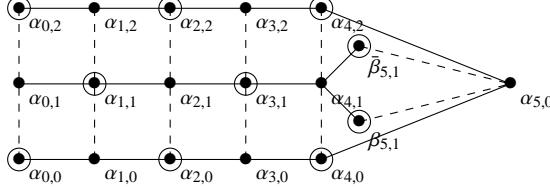
with

$$X = |\chi_{0,0} + \chi_{10,2}|^2 + |\chi_{0,2} + \chi_{10,0}|^2 + |\chi_{2,0} + \chi_{8,2}|^2 + |\chi_{0,2} + \chi_{8,0}|^2 + |\chi_{4,0} + \chi_{6,2}|^2 + |\chi_{4,2} + \chi_{6,0}|^2$$

$$Y = |\chi_{1,1} + \chi_{9,1}|^2 + |\chi_{3,1} + \chi_{7,1}|^2.$$

One can read off the number of irreducible sectors:  $|_N \Delta_N| = 33$ ,  $|_N \Delta_M| = |_M \Delta_M^\pm| = 18$ ,  $|_M \Delta_M| = 36$  and  $|_M \Delta_M^0| = 10$ . The category  ${}_N \mathcal{C}_N$  has  $A_{11} \times A_3$  fusion rules, see Figure 1 and the  $\mathbb{Z}_2$ -simple current extension is an “orbifold” giving the fusion rules of  ${}_M \mathcal{C}_M^\pm$ , Figure 3.

**Example 4.12** ( $E_8$ -cases). Note the net  $\mathcal{B}_{N \subset M}$  for the  $E_8$  subfactor can be realized as  $\mathcal{A}_{D_{16}} \otimes \mathcal{A}_{F_{4,1}}$ , where we can replace  $\mathcal{A}_{F_{4,1}}$  by the  $\tilde{\mathcal{B}} \supset \tilde{\mathcal{A}}_{28}$  the extension from Proposition 4.1 of  $\mathcal{B} = \mathcal{A}_{G_{2,1}} \supset \tilde{\mathcal{A}}_{28}$ . The modular invariant of the inclusion  $\mathcal{A}_{28} \otimes \mathcal{A}_{F_{4,1}} \subset \mathcal{A}_{D_{16}} \otimes \mathcal{A}_{F_{4,1}}$  is  $Z_{D_{12}} \otimes I_2$ .

FIGURE 3. The fusion graph of  ${}_M\mathcal{C}_M^+$  for  $\mathcal{A}_{\text{SU}(2),10} \otimes \mathcal{A}_{\text{Spin}(11),1} \subset \mathcal{A}_{N \subset M}$  for  $E_6$ 

## 5. CATEGORICAL PICTURE AND VERTEX OPERATOR ALGEBRAS

Local irreducible extensions  $\mathcal{B} \supset \mathcal{A}$  of completely rational nets are characterized by commutative Q-systems  $\Theta \in \text{Rep}(\mathcal{A})$  [LR95] and the representation theory is given by the ambichiral sectors  ${}_M\mathcal{C}_M^0$ . The Q-system is a commutative (Frobenius) algebra in the braided tensor category  $\text{Rep}(\mathcal{A})$ . Because  $\Theta$  is commutative, the right-modules  $\text{Mod}(\Theta) = \mathcal{C}_\Theta$ , see [KO02], form itself a tensor category. This category is equivalent with  ${}_M\mathcal{C}_M^+$ . Interchanging the braiding, there is another tensor product under which  $\text{Mod}(\Theta)$  is equivalent with  ${}_M\mathcal{C}_M^-$ . The ambichiral sectors are braided equivalent  ${}_M\mathcal{C}_M^0$  with the category of local or dyslexic modules  $\text{Mod}_0(\Theta)$ , see [BKL14].

The same categorical structure arises for extensions of vertex operator algebras. [KO02, HKL14]. It follows:

**Proposition 5.1.** *Let  $\mathcal{A}$  be a completely rational conformal net and  $V$  a vertex operator algebra, such that the category  $\mathcal{C}_V$  has a natural vertex tensor category structure (cf. [HKL14]) and is braided equivalent to  $\text{Rep}(\mathcal{A})$ . Then for every local irreducible extension  $\mathcal{B} \supset \mathcal{A}$  there exists a vertex operator algebra  $V_{\mathcal{B}} \supset V$ , whose category of modules is braided equivalent to  $\text{Rep}(\mathcal{B})$ .*

Using this proposition we can transport our result to vertex operator algebras. By [FK93, Proposition 8.2.6] ribbon categories with  $\text{SU}(2)_k$  are determined by its twists which are given by the exponential of the conformal weights using [GL96]. The fusion rules calculated by [Was98] coincide with the one of the corresponding affine Kac–Moody VOA. Thus we can conclude that the modular tensor categories are equivalent.

For a VOA corresponding to the net  $\mathcal{A}_k$ , i.e. a VOA which has the opposite braiding of  $\text{SU}(2)_k$ , we could in principle apply Proposition 5.1, but we do not know that the categories for the Virasoro minimal models are equivalent for VOAs and conformal nets.

But we can argue as follows. Let  $V_k = V_{\text{SU}(2)_k}$  be the vertex operator algebra of affine Kac–Moody algebra  $\hat{\mathfrak{sl}}_2$  at level  $k$ . As in Proposition 4.4 we get an inclusion into  $V_{E_8}^{\otimes k}$ , where  $V_{E_8}$  is the vertex operator algebra associated with the even lattice  $E_8$ , which coincides by the Kac–Frenkel construction with the affine Kac–Moody algebra of the Lie algebra  $E_8$  at level 1. Let  $\tilde{V}_k$  be the coset of the inclusion  $V_k \subset V_{E_8}^{\otimes k}$ . Then  $V_{E_8}^{\otimes k}$  decomposes as

$$\bigoplus Z_{kl} M_k \otimes \tilde{M}_l,$$

where  $M_k$  are modules of  $V_k$  and  $\tilde{M}_l$  of the coset net  $\tilde{V}_l$ . It is  $Z_{k0} = \delta_{k,0}$  and  $Z_{0l} = \delta_{l,0}$ . We call such an inclusion of  $V_k \subset V_{E_8}^{\otimes k}$  normal. By the same argument as in Proposition 5.1 the analogue of Proposition 4.3 holds using the same proof and  $\tilde{V}_k$  has as representation category  $\text{SU}(2)_k$  with the opposite braiding.

Then Corollary 4.7 together with Proposition 5.1 gives:

**Proposition 5.2.** *There is a unitary rational VOA  $\tilde{V}_k$  which has the opposite braiding of  $\text{SU}(2)_k$ .*

For every subfactor  $[M : N] < 4$  there is a unitary rational VOA  $V_{N \subset M}$ , whose category of modules is equivalent to the quantum double  $D(N \subset M)$  of the subfactor  $N \subset M$ , i.e. the Drinfeld center of the fusion category of the even part of  $N \subset M$ .

**Remark 5.3.** For the construction of  $\tilde{V}_k$  and  $V_{N \subset M}$  we could also use directly the correspondence between conformal nets and vertex operator algebras in [CKLW15]. We still have to use the categorical arguments to show that the corresponding representations categories are equivalent. It would be nice to have a result that states that the representation categories of  $V$  and  $\mathcal{A}_V$  are the same.

**Example 5.4.** Let  $V$  be the vertex operator algebra obtained by  $\mathbb{Z}_2$ -simple current extension  $\hat{\mathfrak{sl}}_{2,10} \otimes \hat{\mathfrak{so}}_{11,1}$ . Then the category modules of  $V$  is equivalent to  $Z(\frac{1}{2}E_6)$ , the quantum double of the  $E_6$  subfactor.

## 6. CONCLUSIONS AND OUTLOOK

We gave some structural results of completely rational conformal nets whose representation category is a quantum double (Drinfeld center of a unitary fusion category). We showed that the quantum doubles of subfactors with index less than 4, or equivalently the Drinfeld centers of their even part fusion categories, are realized as representation theories in chiral conformal field theory, either as a conformal net of von Neumann algebras or as VOAs. The most interesting is the realization of the quantum double of  $E_6$  (or  $\bar{E}_6$ ) as a  $\mathbb{Z}_2$ -simple current extension of  $\mathrm{SU}(2)_{10} \times \mathrm{Spin}(11)_1$ . In particular, [HRW08] it was shown that the quantum double of  $E_6$  is universal for topological quantum computing. On the other hand, it was proposed in the same article that it might be exotic. Our construction shows that it is indeed not exotic. This example was the main motivation of the article, because no direct realization in conformal field theory or quantum groups is contained in the literature. Further, the even part of  $E_6$  is the smallest non-trivial fusion category [Ost13] in the sense that it is not braided or coming from groups. Drinfeld centers of braided fusion categories and groups are easy. Despite the fact that the even part of  $E_6$  has no braiding, the realization as a CFT is still very easy.

We conjecture that the double of  $E_6$  is also related to Chern–Simons theory with non-simply connected gauge group  $(\mathrm{SU}(2) \times \mathrm{Spin}(11))/\mathbb{Z}_2$ . It is also related to the  $\mathrm{SU}(2)_{10} \times \mathrm{Spin}(11)_1$  quantum group as a kind of quantum subgroup. Indeed the  $\mathbb{Z}_2$ -simple current extension correspond to a quantum subgroup in the sense of Ocneanu [Ocn01].

It would be interesting to find realizations of the doubles of exotic subfactors, like the Haagerup subfactor using similar methods like here.

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